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Stratification of the orbit space in gauge theories: the role of nongeneric strata

R Vilela Mendes

Grupo de Física Matemática, Complexo Interdisciplinar, Universidade de Lisboa,
Avenue Gama Pinto, 2-1649-003 Lisboa, Portugal

E-mail: vilela@cii.fc.ul.pt

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Abstract

Gauge theory is a theory with constraints and, for that reason, the space of physical states is not a manifold but a stratified space (orbifold) with singularities. The classification of strata for smooth (and generalized) connections as well as the formulation of the physical space as the zero set of a momentum map, is reviewed. Several important features of nongeneric strata are discussed and new results are presented suggesting an important role for these strata as concentrators of the measure in ground state functionals and as a source of multiple structures in low-lying excitations.

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1. Introduction

There is increasing evidence that gauge theories are Nature's favourite trick. They have led to a number of questions and some answers of interest to both physicists and mathematicians. Factorization by local gauge transformations induces non-trivial bundle structures in gauge theory and, rather than being a smooth manifold, the gauge orbit space is a stratified space. It has an open dense generic stratum and several nongeneric strata. The generic stratum was extensively studied and led to a geometrical understanding of the Gribov ambiguity [1, 2], the Faddeev–Popov technique [3] and anomalies [4]. In contrast, the role of nongeneric strata has not yet been fully clarified (see, however, [5–8]).

In a field theory, physical states are quantum fluctuations around classical solutions and physical processes are path integrals on the space of field configurations. Therefore, because of its full measure, the (generalized) generic connections play the main role in the quantum fluctuations and in the path integral. However, for the classical solutions around which quantum fluctuations take place, there is no reason why they cannot be taken from the nongeneric strata. In fact, the perturbative vacuum is as nongeneric as it could possibly be.

The main concern in this paper is the role that nongeneric strata play in the construction of physical states. In sections 2 and 3 some results are collected concerning the stratification of gauge orbit spaces and the characterization of the physical space as the zero set of a momentum map. Most of these results are widely dispersed in the mathematical literature, sometimes hidden behind considerable formalism. Hence, a short unified summary, as presented in these sections, might be useful.

After a general discussion of possible roles for nongeneric strata, evidence is presented in section 4 for their role in the structure of the ground state measure and low-lying excitations in $SU(2)$ and $SU(3)$ gauge theories.

The discussion of the possible role of nongeneric strata on gauge theories is based on a non-perturbative approximation of the ground state functional. From this approximation, derived by functional integral techniques, a consistent quantum theory can be built [9]. Whether it is qualitatively equivalent to the full $SU(2)$ and $SU(3)$ gauge field theories is an open question. Nevertheless, it provides an infinite-dimensional example of the concentration of the measure near nongeneric strata, a feature that in the past had only been obtained in finite-dimensional toy models [6, 10].

2. Stratification of the orbit space in gauge theories

A classical gauge theory consists of four basic objects:

- (i) A principal fibre bundle $P(M, G)$ with structural group G and projection $\pi : P \rightarrow M$, the base space M being an oriented Riemannian manifold.
- (ii) An affine space \mathcal{C} of connections ω on P , modelled by a vector space \mathcal{A} of 1-forms on M with values on the Lie algebra \mathcal{G} of G .
- (iii) The space of differentiable sections of P , called the *gauge group* \mathcal{W}
- (iv) A \mathcal{W} -invariant functional (the Lagrangian) $\mathcal{L} : \mathcal{A} \rightarrow \mathbb{R}$.

Choosing a reference connection, the affine space of connections on P may be modelled by a vector space of \mathcal{G} -valued 1-forms ($C^\infty(\Lambda^1 \otimes \mathcal{G})$). Likewise the curvature F is identified with an element of ($C^\infty(\Lambda^2 \otimes \mathcal{G})$).

In coordinates one writes

$$A = A_\mu^a dx^\mu t_a \quad x \in M \quad t_a \in \mathcal{G}$$

and the action of $\gamma = \{g(x)\} \in \mathcal{W}$ on A is given by

$$\gamma : A_\mu(x) \rightarrow (gA_\mu)(x) = g(x)A_\mu(x)g^{-1}(x) - (\partial g)(x) \cdot g^{-1}(x). \quad (1)$$

All statements below refer to the case where G is a compact group.

The action of \mathcal{W} on \mathcal{A} leads to a stratification of \mathcal{A} corresponding to the classes of equivalent *orbits* $\{gA; g \in \mathcal{W}\}$. Let S_A denote the *isotropy (or stabilizer) group* of $A \in \mathcal{A}$,

$$S_A = \{\gamma \in \mathcal{W} : \gamma A = A\}. \quad (2)$$

The *stratum* $\Sigma(A)$ of A is the set of connections having isotropy groups \mathcal{W} -conjugated to that of A ,

$$\Sigma(A) = \{B \in \mathcal{A} : \exists \gamma \in \mathcal{W} : S_B = \gamma S_A \gamma^{-1}\}. \quad (3)$$

The *configuration space of the gauge theory* is the quotient space \mathcal{A}/\mathcal{W} and therefore a stratum is the set of points in \mathcal{A}/\mathcal{W} that correspond to orbits with conjugated isotropy groups.

The stratification of the gauge space when G is a compact group has been extensively studied [11–16]. The stratification is topologically regular. The map that, to each orbit, assigns the conjugacy class of its isotropy group is called the *type*. The set of strata carries a partial

ordering of types, $\Sigma_\tau \subseteq \Sigma_{\tau'}$ with $\tau \leq \tau'$ if there are representatives S_τ and $S_{\tau'}$ of the isotropy groups such that $S_\tau \supseteq S_{\tau'}$. The maximal element in the ordering of types is the class of the centre $Z(G)$ of G and the minimal one is the class of G itself. Furthermore $\cup_{t \geq \tau} \Sigma_t$ is open and Σ_τ is open in the relative topology in $\cup_{t \leq \tau} \Sigma_t$.

Most of the stratification results have been obtained in the framework of Sobolev connections and Hilbert Lie groups. However, for the calculation of physical quantities in the path integral formulation

$$\langle \phi \rangle = \int_{\mathcal{A}/\mathcal{W}} \phi(\xi) e^{i\mathcal{L}(\xi)} d\mu(\xi) \tag{4}$$

a measure in \mathcal{A}/\mathcal{W} is required, and no such measure has been found for Sobolev connections. Therefore, it is more convenient to work in a space of *generalized connections* $\overline{\mathcal{A}}$, defining parallel transports on piecewise smooth paths as simple homomorphisms from the paths on M to the group G , without a smoothness assumption [17]. The same applies to the generalized gauge group $\overline{\mathcal{W}}$. Then, there is in $\overline{\mathcal{A}}/\overline{\mathcal{W}}$ an induced Haar measure, the Ashtekar–Lewandowski measure [18, 19]. Sobolev connections are a dense zero measure subset of the generalized connections [20]. The question remained however of whether the stratification results derived in the context of Sobolev connections would apply to generalized connections. This question was recently settled by Fleischhack [21] who, by establishing a slice theorem for generalized connections, proved that essentially all existing stratification results carry over to the generalized connections. In some cases they even have wider generality.

Because the isotropy group of a connection is isomorphic to the centralizer of its holonomy group [22], the strata are in one-to-one correspondence with the Howe subgroups of G , that is, the subgroups that are centralizers of some subset in G . Given a holonomy group H_τ associated with a connection A of type τ , the stratum of A is classified by the conjugacy class of the isotropy group S_τ , that is, the centralizer of H_τ

$$S_\tau = Z(H_\tau). \tag{5}$$

An important role is also played by the centralizer of the centralizer

$$H'_\tau = Z(Z(H_\tau)) \tag{6}$$

that contains H_τ itself. If H'_τ is a proper subgroup of G , the connection A reduces locally to the subbundle $P_\tau = (M, H'_\tau)$. Global reduction depends on the topology of M , but it is always possible if P is a trivial bundle. H'_τ is the structure group of the *maximal subbundle* associated with type τ . Therefore the types of strata are also in correspondence with the types of reductions of the connections to subbundles. If S_τ is the centre of G the connection is called *irreducible*, all others are called *reducible*. The stratum of the irreducible connections is called the *generic stratum*. It is open and dense and it carries the full Ashtekar–Lewandowski measure.

3. Constraints and momentum maps

3.1. Singularity structure of Yang–Mills solutions: linear and quadratic constraints

The canonical formulation is the more appropriate one to discuss the role of nongeneric strata in gauge theories. This is because it allows a clear separation between gauge invariance and the role of constraints. It uses the theory of bifurcations of zero level sets of momentum mappings as developed by Arms [23, 24] and Arms *et al* [25]. The main points of this construction are summarized below (using an explicit coordinatewise notation).

With $A^\mu = A_a^\mu t_a$ ($\{t_a\}$ a basis for the Lie algebra), one takes

$$\begin{aligned} A_a^i \\ E_a^i = F_a^{i0} = \partial^i A_a^0 - \partial^0 A_a^i - f_{bca} A_b^i A_c^0 \end{aligned} \quad (7)$$

as canonical variables. If \mathcal{A} is the set of vector potentials in M (M is a compact spacelike Cauchy surface or the 3-plane $x^0 = 0$ with appropriate decaying conditions on the fields at infinity) the set (A, E) is a phase-space coordinate in the cotangent bundle of \mathcal{A} :

$$(A, E) \in T^*\mathcal{A} \equiv (C^\infty(\Lambda^1 \otimes \mathcal{G}), C^\infty(\Lambda^2 \otimes \mathcal{G})). \quad (8)$$

Write the Yang–Mills first-order action as

$$I = \frac{2}{g^2} \int d^4x \operatorname{Tr} \left\{ \partial^0 A \cdot E + \frac{1}{2}(E^2 + B^2) - A^0(\nabla \cdot E + [A, E]) \right\} \quad (9)$$

with $B_a = -\frac{1}{2}\epsilon_{ijk} F_a^{jk}$. Then the Hamiltonian is

$$H = \int d^3x \sum_a (E_a^2 + B_a^2) \quad (10)$$

and A^0 being a Lagrange multiplier, the constraint is

$$\Gamma(x) = \nabla \cdot E + [A, E] = D \cdot E = 0. \quad (11)$$

This being a well-posed Cauchy problem, to characterize the singularities of the solutions it suffices to characterize the singularities of the constraint equations.

With canonical brackets

$$\{A_a^i(x), E_b^j(y)\}_{x^0=y^0} = \delta^{ij} \delta_{ab} \delta^3(x-y) \quad (12)$$

one obtains for the infinitesimal gauge transformations

$$\begin{aligned} \delta E_a^i(x) &= f_{bca} \delta \alpha_b(x) E_c^i(x) \\ &= \int d^3y \left\{ E_a^i(x), \sum_b \delta \alpha_b(y) \Gamma_b(y) \right\}_{x^0=y^0} \end{aligned} \quad (13)$$

and

$$\begin{aligned} \delta A_a^i(x) &= -\partial^i \delta \alpha_a(x) + f_{bca} \delta \alpha_b(x) A_c^i(x) \\ &= \int d^3y \left\{ A_a^i(x), \sum_b \delta \alpha_b(y) \Gamma_b(y) \right\}_{x^0=y^0}. \end{aligned} \quad (14)$$

Therefore $\Gamma_b(x)$ behaves as a Hamiltonian function for the flow corresponding to the group element generated by t_b . Hence,

$$(A, E) \in T^*\mathcal{A} \xrightarrow{J} D \cdot E \in C^\infty(\Lambda^3 \otimes \mathcal{G}) \quad (15)$$

is what is called a *momentum mapping* for the symmetry group. This mapping will be denoted by J . Because $C^\infty(\Lambda^3 \otimes \mathcal{G})$ is dual to $C^\infty(\Lambda^0 \otimes \mathcal{G}^*)$ this mapping may also be considered as a mapping from $T^*\mathcal{A}$ to $h^* = C^\infty(\Lambda^0 \otimes \mathcal{G}^*)$ (the space of smooth sections on the Lie algebra dual),

$$(A, E) \xrightarrow{J} \Gamma_b(x) t^b \quad (16)$$

with $\{t^b\}$ as a basis for the dual Lie algebra \mathcal{G}^* .

The constraint $D \cdot E = 0$ means that the set of solutions of Yang–Mills theory is the zero set of a momentum mapping.

For the characterization of the set of solutions of the constraint equations an important role is played by the derivative mapping $J' : T_{(A,E)}(T^*\mathcal{A}) \rightarrow h^*$ and its adjoint $J'^* : h^* \rightarrow T_{(A,E)}(T^*\mathcal{A})$. Linearizing A and E around a background field (\bar{A}, \bar{E})

$$A = \bar{A} + a \quad E = \bar{E} + e \quad (17)$$

one easily obtains

$$\begin{aligned} J'(a, e)_b &= \partial_i e_b^i + f_{bca}(\bar{A}_c^k e_a^k + a_c^k \bar{E}_a^k) \\ (J'^* v)_a^k &= (f_{bac} \bar{E}_c^k v_b(x), (\partial^k \delta_{ba} + f_{bca} \bar{A}_c^k) v_b(x)). \end{aligned} \quad (18)$$

Using pointwise metrics on M and \mathcal{G} and integration, a Riemannian structure is defined in $T^*\mathcal{A}$

$$\langle (a1, e1), (a2, e2) \rangle = \int d^3x (a1a2 + e1e2) \quad (19)$$

which is related to the symplectic form by the complex structure

$$\mathbb{J}(a1, e1) = (-e1, a1) \quad (20)$$

$$\omega((a1, e1), (a2, e2)) = \langle \mathbb{J}(a1, e1), (a2, e2) \rangle. \quad (21)$$

Because J'^* is elliptic with injective principal symbol (in \mathcal{L}^2), one has the \mathcal{L}^2 splittings

$$\begin{aligned} T_{(\bar{A}, \bar{E})}(T^*\mathcal{A}) &= \text{Ker } J' \oplus \text{Im } J'^* \\ &= \text{Im}(\mathbb{J} \circ J'^*) \oplus \text{Ker}(J' \circ \mathbb{J}) \end{aligned} \quad (22)$$

$$h^* = \text{Ker } J'^* \oplus \text{Im } J'. \quad (23)$$

One denotes by

- P , the projection $T^*\mathcal{A} \rightarrow \text{Im } J'$
- H , the projection $T^*\mathcal{A} \rightarrow \text{Ker } J'^*$.

Elimination of redundant variables (or gauge fixing) corresponds, in geometrical terms, to the construction of a slice for the action of the gauge group \mathcal{W} . A slice through a point $(\bar{A}, \bar{E}) \in T^*\mathcal{A}$ is a submanifold $S \subset T^*\mathcal{A}$ such that

- (i) $\gamma(\bar{A}, \bar{E}) = (\bar{A}, \bar{E}) \implies \gamma S = S, \quad \gamma \in \mathcal{W}$
- (ii) $\gamma S \cap S \neq \emptyset \implies \gamma(\bar{A}, \bar{E}) = (\bar{A}, \bar{E})$
- (iii) $T^*\mathcal{A}$ is locally the product of the slice S and the orbit of (\bar{A}, \bar{E}) .

In this setting, the following important results have been obtained [23–25]:

- (1) An orthogonal slice for the group action is

$$(\bar{A}, \bar{E}) + \text{Ker}(J' \circ \mathbb{J}). \quad (24)$$

The orthogonal complement, $\text{Im}(\mathbb{J} \circ J'^*)$, is the tangent space to the orbit at (\bar{A}, \bar{E}) .

- (2) Denote by $C = \{J = 0\}$ the solution set of the constraint equations and

$$C_P = \{P \circ J = 0\} \quad C_H = \{H \circ J = 0\} \quad (25)$$

with $C = C_P \cap C_H$.

Then, there is a smooth mapping (the Kuranishi transformation) that maps C_P locally onto $(\bar{A}, \bar{E}) \oplus \text{Ker } J'$. ($\text{Ker } J'$ is the set of solutions of the linearized constraints, $\partial_i e_b^i + f_{bca}(\bar{A}_c^k e_a^k + a_c^k \bar{E}_a^k) = 0$.)

- (3) $\text{Ker } J'^*$, that is, $(f_{bac}\overline{E}_c^k v_b(x) = 0, (\partial^k \delta_{ba} + f_{bca}\overline{A}_c^k)v_b(x) = 0)$, is the set of infinitesimal symmetries of $(\overline{A}, \overline{E})$. If $\text{Ker } J'^* = 0$, then $C_P = C$ and in this case the solution set of the full constraint equations is a manifold near $(\overline{A}, \overline{E})$ with tangent space $\text{Ker } J'$. It means that, if the background has no symmetries, any solution of the linearized equations approximates to first-order, a curve of exact solutions.
- (4) In case the background has non-trivial symmetries ($\text{Ker } J'^* \neq 0$), define a set QC as follows,

$$QC = \{(a, e) \in T_{(\overline{A}, \overline{E})}(T^* \mathcal{A}) : (a, e) \in (\text{Ker}(J' \circ \mathbb{J}) \cap \text{Ker } J') \text{ and } [a \wedge e] = 0\} \quad (26)$$

in coordinates

$$\begin{aligned} \partial_i a_b^i + f_{bca}(\overline{A}_c^k e_a^k + a_c^k \overline{E}_a^k) &= 0 \\ \partial_i e_b^i + f_{bca}(\overline{A}_c^k a_a^k - e_c^k \overline{E}_a^k) &= 0 \\ [a \wedge e]_b = f_{bca} a_c^k(x) e_a^k(x) &= 0. \end{aligned} \quad (27)$$

The first condition is the (gauge fixing) condition that restricts the perturbation to the slice. The second is the linearized constraint and the third a quadratic constraint condition.

Then, S being the slice, there is a local diffeomorphism (Kuranishi's) of $C \cap S$ onto $(\overline{A}, \overline{E}) + QC$. Equivalently, the non-linear constraint set is locally $C \approx \text{Orbit}(\overline{A}, \overline{E}) \oplus QC$.

It means that, when the background has non-trivial symmetries, there are solutions of the linearized equations that are not tangent to actual solutions and a further quadratic constraint must be imposed on the perturbations.

- (5) The term $[a \wedge e]_b = f_{bca} a_c^k(x) e_a^k(x)$, used above, is the diagonal of the quadratic form

$$J''((a1, e1), (a2, e2)) = [a1 \wedge e2] + [a2 \wedge e1]. \quad (28)$$

The degeneracy space of this quadratic form characterizes the solutions with the same symmetries as $(\overline{A}, \overline{E})$, namely, as follows.

The set of solutions with the same symmetries as $(\overline{A}, \overline{E})$ is a manifold with tangent space at $(\overline{A}, \overline{E})$ given by

$$\{(a, e) : (a, e) \in (\text{Ker}(J' \circ \mathbb{J}) \cap \text{Ker } J') \text{ and } J''((a, e), (a1, e1)) = 0\} \quad (29)$$

for all $(a1, e1) \in (\text{Ker}(J' \circ \mathbb{J}) \cap \text{Ker } J')$.

The results listed above have some practical consequences. They mean, for example, that in perturbative calculations around a background $(\overline{A}, \overline{E})$ with non-trivial symmetries, linear perturbations must be further restricted by a quadratic condition. In quantum perturbation theory, the quadratic condition becomes an operator condition and physical perturbations must be annihilated by the corresponding quadratic operator.

Further consequences and roles for the nongeneric backgrounds are explored in the remainder of the paper.

3.2. Confinement and the singlet structure of excitations

The confinement question covers two distinct statements:

- (i) All observables are colour neutral.
- (ii) All physical states are colour singlets.

For fields transforming under a non-Abelian gauge group, once it is assumed that the gauge invariance is an exact symmetry, the first statement is a simple manifestation of the existence of a non-Abelian superselection rule. This has been proved long ago by Strocchi [26]. Let $\{Q_a\}$ be the set of colour charges that generates the (global) gauge group G and \mathcal{O} a local observable. Computing the commutator between physical states ψ and ϕ ,

$$\langle \psi | [Q_a, \mathcal{O}] | \phi \rangle = \langle \psi | \left[\int d^3x J_a^0, \mathcal{O} \right] | \phi \rangle = \langle \psi | \left[\int d^3x (J_a^0 - \partial_i F_a^{i0}), \mathcal{O} \right] | \phi \rangle = 0 \quad (30)$$

where the second equality follows from locality of \mathcal{O} and the third from Gauss' law

$$\partial_i F_a^{i0} = j_a^0 + g f_{abc} A_{ib} F_c^{i0} = J_a^0 \quad (31)$$

acting on physical states. The term j_a^0 denotes the non-gluonic charge.

Equation (30) implies that all local observables, in the physical space, commute with the colour charges, that is, G is a non-Abelian superselection rule. In particular, it implies that (local) colour charges cannot be observable quantities. Therefore the fact that colour charges are not observable is not a dynamical question, in the sense that it does not depend on the detailed dynamics of non-Abelian gauge theory but simply on the fact that the current conservation occurs in a particular form, namely the current is the divergence of an antisymmetric tensor. Unobservability of colour charges is therefore a trivial consequence of non-Abelian gauge symmetry. The deep question is of course why there is an exactly conserved colour gauge symmetry.

It is possible that the second of the confinement statements—the existence of just colour singlets—may also be a ‘kinematical’ consequence of gauge symmetry, although the situation here is not so obvious. The existence of a non-Abelian superselection rule implies that the superselection sectors are labelled by the eigenvalues of the Casimir operators. For all except the singlet sector, there will be more than one vector corresponding to the same physical state. Hence if non-singlet states were to exist, their description would imply a departure from the usual quantum mechanical framework. Namely, there would not exist a complete commuting set of observables and the description of scattering experiments, for example, would require special care because the computed matrix elements would depend on the initial and final vector representatives chosen among the physically equivalent multiplet vectors [27]. A description using direct integral spaces [28] or some other form of averaging over initial and final physically equivalent vectors would be mandatory to obtain unambiguous predictions.

If colour is an unbroken symmetry, the question of confinement is not whether any coloured states are going to be found, because colour charges are unobservable anyway, but whether the colourless objects one sees are real singlets or some sort of balanced admixture of hidden colour states. It is here, however, that nongeneric strata play a role. The quadratic condition $f_{bca} a_c^k(x) e_a^k(x) = 0$ means exactly that the (gluonic) charge associated with excitations around a nongeneric background is zero. Therefore if the background belongs to a nongeneric stratum, the perturbative vacuum for example, the gluonic low-lying excitations around this background must be singlets. Nothing is said, of course, concerning excitations around generic states or non-gluonic states.

3.3. Suppression of non-symmetric fluctuations and wavefunctional enhancements

For quantized gravitational fluctuations around a symmetric background spacetime, it has been found [29] that the effect of quadratic constraints is to suppress transitions to configurations of lower symmetry. This led some authors [10, 6] to conjecture that the amplitude of the Schrödinger functional would display particular enhancements (or suppressions) near the

singularities. This was illustrated by studying finite-dimensional examples of the Schrödinger equation in configuration spaces with conical singularities.

However, for gauge theories, there is not always suppression of fluctuations to configurations of lower symmetry. The degeneracy space of the quadratic form J'' (equation (28)) characterizes the fluctuations with the same symmetry as the background (\bar{A}, \bar{E}) . But, in addition to this manifold with the same symmetry as (\bar{A}, \bar{E}) , there are other solutions, with different symmetry, leading to the conical singularity. For an initial condition in some stratum, it is known that classical solutions remain in the same stratum [30], but quantum fluctuations will, in principle, explore all the solutions compatible with the linear and quadratic constraints. Therefore, the conjecture of enhancement near conical singularities may indeed be true, but it does not necessarily follow from the theory described above.

In the next section, by studying an approximation to the ground state functional of $SU(2)$ and $SU(3)$ gauge theories, one finds additional circumstantial evidence for enhancements near particular classes of nongeneric strata.

4. Nongeneric strata in $SU(2)$ and $SU(3)$ gauge theories

4.1. Ground state functionals in gauge theories

Using an approximation to the ground state functional, it will be found that some field configurations, corresponding to reducible strata, concentrate on the ground state measure. The approximation to the ground state functional is based on an expansion of the path integral representation [9]¹

$$|\psi_0(\chi)|^2 = \frac{1}{N} \int \mathcal{D}\chi(\tau) \delta(\chi(0) - \chi) \exp\left(\int_{-\infty}^{\infty} L_E(\chi(\tau), \frac{d}{dt}\chi(\tau)) d\tau\right) \quad (32)$$

where χ denotes the finite- or infinite-dimensional set of configuration space variables, and L_E is the Euclidean action. Making the change of variables

$$\chi(\tau) = \chi + z(\tau) \quad (33)$$

adding a term $z(\tau).J(\tau)$ to the Euclidean Lagrangian $L_E = -c\left(\frac{d}{dt}\chi\right)^2 + V(\chi)$, separating the terms quadratic or less than quadratic in $z(\tau)$ from higher order terms and computing the Gaussian integrals for the fluctuations around each configuration χ , the following representation is obtained for $|\psi_0(\chi)|^2$,

$$|\psi_0(\chi)|^2 = \frac{\exp\left(\int d\tau G\left(\frac{\partial}{\partial J(\tau)}\right)\right) \exp\left(\frac{1}{4} \int d\tau (\Gamma(\chi) - J(\tau)) \frac{1}{-c \frac{\partial^2}{\partial \tau^2} + S(\chi)} (\Gamma(\chi) - J(\tau)) \exp(2\sqrt{c}L(\chi)\sqrt{S(\chi)}L(\chi))\right)}{\sqrt{\det \frac{1}{4\sqrt{c}S(\chi)}} \exp\left(\int d\tau G\left(\frac{\partial}{\partial J(\tau)}\right)\right) \exp\left(\frac{1}{4} \int d\tau (\Gamma(\chi) - J(\tau)) \frac{1}{-c \frac{\partial^2}{\partial \tau^2} + S(\chi)} (\Gamma(\chi) - J(\tau))\right)} \Bigg|_{J=0} \quad (34)$$

where

$$L = -\frac{i}{4\sqrt{c}} \int d\tau \frac{1}{\sqrt{S(\chi)}} \exp\left(-|\tau| \sqrt{\frac{S(\chi)}{c}}\right) (\Gamma(\chi) - J(\tau)) \quad (35)$$

¹ Reference [9] uses a path integral construction of the ground state functional. A more rigorous derivation, based on white noise analysis, leads to an equivalent result [36].

the evaluation of the Gaussian integrals requiring $S(\chi) > 0$. In the $J \rightarrow 0$ limit, L reduces to

$$L_0 = -\frac{i}{2} \frac{1}{S(\chi)} \Gamma(\chi). \tag{36}$$

Expanding $\exp(\int d\tau G(\frac{\partial}{\partial J(\tau)}))$, successive approximations to the ground state are obtained. Of particular interest is the leading term

$$|\psi_0(\chi)|_{(0)}^2 = \left(\det \frac{1}{4\sqrt{cS(\chi)}}\right)^{-\frac{1}{2}} \exp\left\{-\frac{\sqrt{c}}{2} \Gamma(\chi) \frac{1}{S(\chi)} \sqrt{S(\chi)} \frac{1}{S(\chi)} \Gamma(\chi)\right\} \tag{37}$$

which differs from a perturbative estimate in the fact that, around each point of the wavefunctional, a different expansion point is chosen, which is χ itself. In the functional integral representation of the ground state, the wavefunctional is the integrated effect of paths coming from the infinite past to the point χ at $t = 0$. Because near χ the difference $z(\tau) - \chi$ is small, the leading term will contain accurate information from all paths in the neighbourhood of χ , and will be inaccurate only regarding non-harmonic contributions to the paths far away from χ . For many problems, like the quartic or exponential oscillators, one finds that, up to a shift of parameter values, the leading term is already practically indistinguishable from the exact ground state.

For non-Abelian gauge fields one uses the Schrödinger formulation for quantum fields [31, 32]. $\mathcal{A}_i^\alpha(\tau, x)$ is the spacetime Euclidean vector potential, $A_i^\alpha(x) = \mathcal{A}_i^\alpha(0, x)$ the time-zero field, $\mathcal{B}_i^\alpha(\tau, x) = \epsilon_{ijk}(\partial_j \mathcal{A}_k^\alpha - \frac{g}{2} f_{\alpha\beta\gamma} \mathcal{A}_j^\beta \mathcal{A}_k^\gamma)$ the non-Abelian curvature field and $B_i^\alpha(x) = \epsilon_{ijk}(\partial_j A_k^\alpha - \frac{g}{2} f_{\alpha\beta\gamma} A_j^\beta A_k^\gamma)$ its time-zero counterpart. The time-zero fields are the canonical variables and the chromoelectric fields the conjugate momenta. Making the change of variables

$$\mathcal{A}_i^\alpha(\tau, x) = A_i^\alpha(x) + \phi_i^\alpha(\tau, x) \tag{38}$$

the Euclidean Lagrangian is

$$L_E = -\frac{1}{2} \left(\frac{\partial}{\partial \tau} \phi_i^\alpha\right)^2 - \frac{1}{2} (B_i^\alpha)^2. \tag{39}$$

To construct the ground state approximation according to equation (37), note that

$$\mathcal{B}_i^\alpha(\tau, x) = B_i^\alpha(x) + \epsilon_{ijk} D_j(A)_{\alpha\beta} \phi_k^\beta(\tau, x) - \epsilon_{ijk} f_{\alpha\beta\gamma} \phi_j^\beta(\tau, x) \phi_k^\gamma(\tau, x) \tag{40}$$

with

$$D_j(A)_{\alpha\beta} = \partial_j \delta_{\alpha\beta} - g f_{\alpha\beta\gamma} A_j^\gamma(x). \tag{41}$$

Using (37), the leading term for the ground state functional is

$$\psi_0(A)_{(0)} = \exp\left\{-\frac{1}{2} \int d^3x B_k^\alpha(A(x)) \left(\frac{1}{\sqrt{R(A(x)) \cdot R(A(x))}}\right)_{kk'}^{\alpha\alpha'} B_{k'}^{\alpha'}(A)\right\} \tag{42}$$

where the following operator has been defined:

$$R(A)_{nn'}^{\alpha\alpha'} = \epsilon_{nmn'} D_m(A)^{\alpha\alpha'}. \tag{43}$$

4.2. $SU(2)$

If $G = SU(2)$, the isotropy groups and the structure groups of the maximal subbundles are

	S_A	H'_A	
1	\mathbb{Z}_2	$SU(2)$	
2	$U(1)$	$U(1)$	
3	$SU(2)$	\mathbb{Z}_2	(44)

There are three strata. Stratum 1 is the generic stratum. The other two are reducible strata.

For particular classes of fields, the leading ground state approximation, described above, may be given a simple explicit form, which allows us to test the role of the different strata on the construction of low-energy states. Low-energy states are expected to be associated with fields which, at least locally, are slowly varying. Therefore a natural subclass to be studied is that of constant non-Abelian fields $A_i^\alpha(x) = A_i^\alpha$ restricted to a finite space volume V . Consider the matrix

$$M_{ij}^{(2)} = \sum_{\alpha=1}^3 A_i^\alpha A_j^\alpha. \quad (45)$$

Being symmetric, this matrix may be diagonalized by a space rotation. As a result, without loss of generality, $\{A_1^\alpha, A_2^\alpha, A_3^\alpha\}$ is a set of orthogonal vectors and the $SU(2)$ coordinates may be chosen such that

$$A_1^\alpha = (a_1, 0, 0) \quad A_2^\alpha = (0, a_2, 0) \quad A_3^\alpha = (0, 0, a_3). \quad (46)$$

Then

$$B_1^\alpha = -g(a_2 a_3, 0, 0) \quad B_2^\alpha = -g(0, a_3 a_1, 0) \quad B_3^\alpha = -g(0, 0, a_1 a_2). \quad (47)$$

Using a standard representation for the fractional powers of positive operators [33]

$$\frac{1}{\sqrt{R(A) \cdot R(A)}} = \frac{1}{\pi} \int_0^\infty d\lambda \lambda^{-\frac{1}{2}} \frac{1}{\lambda + R(A) \cdot R(A)} \quad (48)$$

and computing

$$B(\lambda + R \cdot R) B^T$$

one obtains

$$\begin{aligned} \psi_0(A)_{(0)} = \exp \left\{ -\frac{Vg}{2\pi} \int_0^\infty d\lambda \lambda^{-\frac{1}{2}} \left[(a_1, a_2, a_3)^2 (a_1^2 + a_2^2 + a_3^2) \right. \right. \\ \left. \left. + \lambda (a_1^4 (a_2^2 + a_3^2) + a_2^4 (a_1^2 + a_3^2) + a_3^4 (a_1^2 + a_2^2)) \right. \right. \\ \left. \left. + \lambda^2 (a_2^2 a_3^2 + a_1^2 a_3^2 + a_1^2 a_2^2) \right] \left[4(a_1 a_2 a_3)^2 + \lambda (\lambda + a_1^2 + a_2^2 + a_3^2)^2 \right]^{-1} \right\}. \quad (49) \end{aligned}$$

This function is peaked at the zeros of the exponent, which only occur when two of the a s vanish. For example, for $a_1 = 0$ the exponent $\sigma(a_1, a_2, a_3)$ in $\psi_0(A)_{(0)} = \exp \left\{ -\frac{Vg}{2} \sigma(a_1, a_2, a_3) \right\}$ becomes

$$\sigma(0, a_2, a_3) = \frac{a_2^2 a_3^2}{\sqrt{a_2^2 + a_3^2}}. \quad (50)$$

As a consequence, for this class of fields, the ground state functional is peaked both near strata of type 2 and 3. Three remarks are in order at this point:

- (i) When two of the constants vanish (for example $a_1 = a_2 = 0$), the chromomagnetic fields in (47) also vanish. Therefore a point strictly on the plane ($a_1 = a_2 = 0$) might be identified by a gauge transformation to the origin. However, by choosing $(a_1 = 0, a_2 = \varepsilon, a_3 = \frac{K}{\varepsilon})$ one obtains, for small ε , $\sigma(0, a_2, a_3) \simeq \varepsilon K$ and $B_1^\alpha = -g(K, 0, 0)$. Therefore arbitrarily large chromomagnetic fields exist with high probability near the plane ($a_1 = a_2 = 0$). The holonomy group is generated by σ_1 with centralizer $U(1)$. This justifies the statement that there is a $U(1)$ stratum acting as a concentrator of the ground state measure.
- (ii) The same reasoning implies that large field fluctuations are to be expected in the non-perturbative vacuum and this approximate ground state functional provides a dynamical view of the vacuum condensates.

(iii) On the other hand, one should not expect all fields in the non-generic strata to act as concentrators of the ground state measure. A counter-example would be a $U(1)$ field with a fast space variation. Therefore, as stated before, a simple reasoning based on suppression of transitions by the quadratic constraints cannot be the whole story.

Similar remarks apply to the role played by the nongeneric strata in $SU(3)$, to be discussed below. That is, in each case, one identifies the subspace where the ground state functional is peaked and then finds the corresponding non-trivial neighbouring stratum, as in (i) above.

4.3. $SU(3)$

For $G = SU(3)$ the isotropy groups and the structure groups of the maximal subbundles are [13] :

	S_A	H'_A	
1	\mathbb{Z}_3	$SU(3)$	
2	$U(1)$	$U(2)$	
3	$U(1) \times U(1)$	$U(1) \times U(1)$	(51)
4	$U(2)$	$U(1)$	
5	$SU(3)$	\mathbb{Z}_3	

There are five strata. Stratum 1 is the generic stratum. All others are reducible strata. Denote by \mathcal{Z} the gauge group transformations with values in the centre, $\tilde{\mathcal{W}} = \mathcal{W}/\mathcal{Z}$ and \mathcal{B} the space (stratum 1) of irreducible connections. Then $\mathcal{B}/\tilde{\mathcal{W}}$ is an open dense set in \mathcal{A}/\mathcal{W} , the complement (the space of reducible connections) being nowhere dense. Most gauge theory studies restrict themselves to $\mathcal{B}/\tilde{\mathcal{W}}$. However, we will see in a while that, like in $SU(2)$, there are important contributions from the nongeneric strata to low-lying states. In addition and contrary to the $SU(2)$ case, this structure is not unique, several possible non-equivalent configurations being possible.

As before one makes a local analysis and considers fields that are constant in a finite volume V . Again, the symmetric matrix

$$M_{ij}^{(3)} = \sum_{\alpha=1}^8 A_i^\alpha A_j^\alpha \tag{52}$$

may be diagonalized by a space rotation. $\{A_1^\alpha, A_2^\alpha, A_3^\alpha\}$ is then a set of three orthogonal vectors in an eight-dimensional space and there are several independent choices. The stratification of the octet space by $SU(3)$ orbits [34], characterizes the independent choices. However, what is important here is not only $SU(3)$ geometrical independence but to characterize the choices that lead to qualitatively different ground state functionals:

(i) Let the only non-zero components be

$$A_1^1 = a_1 \quad A_2^2 = a_2 \quad A_3^3 = a_3. \tag{53}$$

This case is identical to that studied for $SU(2)$, being the measure concentrated now near a stratum with isotropy $U(1) \times U(1)$.

(ii) Let the non-zero components be

$$A_1^1 = a_1 \quad A_2^2 = a_2 \quad A_3^8 = a_8. \tag{54}$$

The only non-zero component of the chromomagnetic field is

$$B_3^3 = -ga_1a_2 \tag{55}$$

and

$$B(\lambda + R \cdot R)B^T = \frac{a_1^2 a_2^2}{\lambda + a_1^2 + a_2^2}. \quad (56)$$

In this case, all the fields belong to a $U(1) \times U(1)$ stratum and the measure being concentrated near $a_1 = 0$ or $a_2 = 0$, with the choice $a_1 = \varepsilon$, $a_2 = \frac{K}{\varepsilon}$ (with ε small) as before, one proves the existence of $U(1) \times U(1)$ fields with large measure. This example shows that not all nongeneric fields are concentrators of the measure. On the other hand, it seems that whenever the measure is peaked, there is a nearby nongeneric stratum field.

(iii) If the non-zero components are

$$A_1^4 = a_4 \quad A_2^5 = a_5 \quad A_3^8 = a_8 \quad (57)$$

$$B_1 = -g \frac{\sqrt{3}}{2} a_5 a_8 \frac{\lambda_4}{2} \quad B_2 = -g \frac{\sqrt{3}}{2} a_4 a_8 \frac{\lambda_5}{2} \quad B_3 = \frac{-g}{2} a_4 a_5 \left(\frac{\lambda_3}{2} + \sqrt{3} \frac{\lambda_8}{2} \right) \quad (58)$$

then

$$B(\lambda + R \cdot R)B^T = 4 \left\{ \begin{aligned} &(16a_4^2 a_5^2 + 12a_5^2 a_8^2 + 12a_4^2 a_8^2) \lambda^2 + (16a_4^4 a_5^4 + 16a_4^4 a_5^2 + 12a_5^4 a_8^2) \\ &+ 9a_5^2 a_8^4 + 12a_4^4 a_8^2 + 9a_4^2 a_8^4 \lambda + 12a_4^4 a_5^2 a_8^2 + 12a_4^4 a_5^4 a_8^2 + 9a_4^2 a_5^2 a_8^4 \end{aligned} \right\} \\ \times \left\{ \begin{aligned} &16\lambda^3 + (32a_5^2 + 32a_4^2 + 24a_8^2) \lambda^2 \\ &+ (16a_4^4 + 16a_5^4 + 9a_8^4 + 32a_4^2 a_5^2 + 24a_8^2 a_4^2 + 24a_5^2 a_8^2) \lambda + 48a_5^2 a_8^2 a_4^2 \end{aligned} \right\}^{-1} \quad (59)$$

and one has the following limits:

for $a_4 = 0$

$$B(\lambda + R \cdot R)B^T = \frac{12a_5^2 a_8^2}{4\lambda + 4a_5^2 + 3a_8^2} \quad (60)$$

for $a_5 = 0$

$$B(\lambda + R \cdot R)B^T = \frac{12a_4^2 a_8^2}{4\lambda + 4a_4^2 + 3a_8^2} \quad (61)$$

for $a_8 = 0$

$$B(\lambda + R \cdot R)B^T = \frac{4a_4^2 a_5^2}{\lambda + a_4^2 + a_5^2}. \quad (62)$$

For $a_4 = a_8 = \varepsilon$ and $a_5 = \frac{K}{\varepsilon}$ (ε small) the holonomy group is $SU(2)$ (V-spin), the centralizer is $U(1)$ (generated by $\sqrt{3}\lambda_3 - \lambda_8$) and, from (62), it follows that this $U(1)$ field is near a point of high measure. If $a_8 = 0$ with the same conditions for a_4 and a_5 , then it would be a field in a $U(1) \times U(1)$ stratum.

(iv) Finally, if

$$A_1 = 0 \quad A_2 = a_2 \left(\frac{\sqrt{2}}{2} \lambda_4 + \frac{\lambda_1}{2} \right) \quad A_3 = a_3 \left(-\frac{\sqrt{2}}{2} \lambda_5 + \frac{\lambda_2}{2} \right) \quad (63)$$

$$B = g \sqrt{3} a_2 a_3 \frac{\lambda_8}{2} \quad (64)$$

then

$$B(\lambda + R \cdot R)B^T = 12 \frac{a_2^2 a_3^2 (4\lambda^2 + 9\lambda(a_2^2 + a_3^2) + 18a_2^2 a_3^2)}{(4\lambda + 3a_2^2 + 3a_3^2)(\lambda^2 + 3\lambda(a_2^2 + a_3^2) + 6a_2^2 a_3^2)}. \quad (65)$$

All fields in this example belong to a stratum with isotropy group $U(2)$. The measure is peaked near $a_2 = 0$ or $a_3 = 0$ and with $(a_2 = \varepsilon, a_3 = \frac{\kappa}{\varepsilon})$ one finds non-trivial $U(2)$ fields near the maximum of the measure.

In conclusion, one sees that there are geometrical independent choices which lead to ground state functionals with measures concentrated near each one of the nongeneric strata 2 to 4. Low-lying physical states being represented by quantum fluctuations around the ground state functional, one also concludes that, at least in the leading-order approximation of the expansion leading to equation (42), there may be distinct classes of excitations around each type of nongeneric strata. This is a much richer structure than that implied by the perturbative vacuum (stratum 5). On the other hand, the high probability that is assigned to large chromomagnetic fluctuations in the ground state functional is consistent with the phenomenological evidence for the existence of non-trivial vacuum condensates in the QCD vacuum [37, 38].

5. Conclusions

By formulating, in the Hamiltonian formalism, the (primary) constraint as the zero set of a momentum map, a clear view is obtained of the dual role of gauge invariance and constraints, as well of the singularity structure of the stratified orbit space in gauge theories. Some physical consequences of these results are the need to impose quadratic constraints on perturbation theory and the natural singlet structure of excitations around nongeneric backgrounds. These are important roles for the nongeneric strata both in classical and quantum theory. In addition, the role of nongeneric strata on the structure of anomalies has been discussed in the past [35].

As a further role for nongeneric strata, there are conjectures concerning enhancements of the lowest lying Schrödinger functional near these strata. The study of finite-dimensional examples with conical singularities provided support for this conjecture [10, 6]. Here, using a non-perturbative approximation to the ground state functional in $SU(2)$ and $SU(3)$ gauge theory, more circumstantial evidence was provided for this conjecture. In addition to the concentration of the measure near nongeneric strata, there is also the possibility of a multiplicity of distinct excitations associated with each stratum type.

Strata of gauge groups G and the structure groups H of subbundles have been studied in the past in the context of symmetry breaking from G to H (see, for example, [39]). Symmetry breaking corresponds to a reduction to a subbundle associated with the subgroup. The possibility of making this reduction depends on the global structure of the base manifold M . This problem is not addressed here, because we have been concerned mostly with a local analysis. Furthermore, in our discussion of non-trivial vacuum backgrounds, no symmetry breaking is implied. All equivalent directions in the functional (42) are equiprobable and the full gauge symmetry is preserved.

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